

Exam II MTH 512 , Fall 2018

Ayman Badawi

34/40

QUESTION 1. Let V, W be nonzero elements of an inner product space X over R. Let $D = W - \frac{<V,W>}{<V,V>}V$

(i) Find $< V, D >$.

$<V,D> = <V, W - \frac{<V,W>}{<V,V>} V> = <V,W> - \frac{<V,W>}{<V,V>} \cancel{<V,V>}$ (over $\mathbb{R}$)

$<V,W> + <V, -\frac{<V,W>}{<V,V>} V>$

$= 0 \in \mathbb{R}$

4/4

(ii) If V, W are dependent (i.e., $W = cV$ for some $c \in R$) , what is D? Can you tell me what is c?

$D = W - \frac{<V,W>}{<V,V>} V = cV - \frac{<V,cV>}{<V,V>} \cdot V = cV - c\frac{<V,V>}{<V,V>} V$

$= cV - cV = 0_X$ (0-vector)

since $D = 0$-vector,
$W = \frac{<V,W>}{<V,V>} V$
$c = \frac{<V,W>}{<V,V>}$

3/4

$W = cV$

$||W|| = ||cV|| = \cancel{|c| \, ||V||} \Rightarrow |c| = \frac{||W||}{||V||} = \frac{||W||}{||V||}$

(iii) Assume $X = R^4$ and $L = (-1, 4, -1, 1) \in span\{V, W, Y = (-2, 2, 2, 2)\}$, where V, W, Y are orthogonal. Then $L = c_1V + c_2W + c_3Y$. Find c_3.

inner product both sides by Y

$<L,Y> = <c_1V + c_2W + c_3Y, Y> = <c_1V, Y> + <c_2W, Y> + <c_3Y, Y>$
$= c_1 <V,Y> + c_2 <W,Y> + c_3 <Y,Y>$
$= 0 + 0 + c_3 <Y,Y>$ (V, W, Y orthogonal)

$c_3 = \frac{<L,Y>}{<Y,Y>} = \frac{10}{16} = \frac{5}{8}$

$<L,Y> = (-1,4,-1,1)\begin{bmatrix} -2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = 2+8-2+2 = 10$

$<Y,Y> = 4+4+4+4 = 16$

4/4

QUESTION 2. (Short) (one line (at most 2 lines) proof). Let $T \in L(R^2, R^3)$. Assume $w \in Range(T^*)$ and $y \in Z(T)$ (note $Z(T)$ is the null space of T. Show that $< y, w >= 0$. (note that $T^*(h) = w$ for some $h \in R^3$)

~~$T \in L(\mathbb{R}^2, \mathbb{R}^3)$, $T^* \in L(\mathbb{R}^3, \mathbb{R}^2)$, $T^* : \mathbb{R}^3 \to \mathbb{R}^2$~~

~~$w \in Range(T^*)$, $y \in Z(T)$~~

~~We know that $Z(T) = Range(T^*)^\perp$~~

~~This means that $y \in Z(T)$ & $w \in Range(T^*)$~~

~~$\Rightarrow <y,w> = 0$~~

Note $<y, T^*(h)> = <T(y), h> = <0, h> = 0$

3/4

see back of the page for proof.

QUESTION 3. Let $T: R^3 \to R^2$ such that $T(a,b,c) = (a+c, -b)$ (you may consider the normal dot product on R^3 and R^2). Find T^*.

Let M be standard matrix representation of T

$T(a,b,c) = M\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\quad \underset{2\times 3}{M} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ $\quad M^* = M^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$

M^* is the s.m.r of $T^*: \mathbb{R}^2 \to \mathbb{R}^3$

$T^*(m,n) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} = m\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + n\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = (m, -n, m)$

(Let $(m,n) \in \mathbb{R}^2$) $\quad \Rightarrow T^*(m,n) = (m, -n, m)$

4/4

QUESTION 4. Let $W = span\{\underset{w_1}{(1,1,0,1)}, \underset{w_2}{(-1,-1,1,1)}\}$. Find the orthogonal complement of W (you may use the dot product)

Let $\underset{2\times 4}{M} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$ be the s.M.R of $T: \mathbb{R}^4 \to \mathbb{R}^2$

$M^* = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ is the s.M.R of $T^*: \mathbb{R}^2 \to \mathbb{R}^4$

$Range(T^*) = span\{(1,1,0,1), (-1,-1,1,1)\} = W$

$\Rightarrow W^\perp = Range(T^*)^\perp = Z(T)$

Solve homogeneous:

$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$

4/4

$x_1 + x_2 + x_4 = 0 \qquad x_3 + 2x_4 = 0$

$x_1 = -x_2 - x_4 \qquad x_3 = -2x_4$

$\Rightarrow W^\perp = \{(-x_2 - x_4, x_2, -2x_4, x_4) \mid x_2, x_4 \in \mathbb{R}\}$

$= \{x_2(-1,1,0,0) + x_4(-1,0,-2,1) \mid x_2, x_4 \in \mathbb{R}\}$

$W^\perp = span\{\underset{v_1}{(-1,1,0,0)}, \underset{v_2}{(-1,0,-2,1)}\}$ ← orthogonal complement of W

Check

$\langle w_1, v_1 \rangle = -1 + 1 = 0$

$\langle w_1, v_2 \rangle = -1 + 1 = 0$

$\langle w_2, v_1 \rangle = 1 - 1 = 0$

$\langle w_2, v_2 \rangle = 1 + 0 - 2 + 1 = 0$

Question 6: continued

$$R_2 \leftrightarrow R_3 \quad \sim \quad \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 \leftrightarrow C_3 \quad \sim \quad \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{C}$$

$|D| = 16 \quad |A| = -16 \quad \& \, a|b|c$

check

$$\underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} 2 & 4 & 2 \\ -2 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}}_{A} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 2 \\ 0 & 4 & 4 \end{bmatrix}}_{RA} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{D} \checkmark$$

$\Rightarrow RAC = D$ ✓

QUESTION 5. Let $W = span\{(1,1,1,0),(-1,0,-1,0)\}$ and $V = span\{(2,0,1,0),(1,0,0,0)\}$. Find a basis for $V + W$. Find a basis for $W \cap V$.

$w_1 \quad w_2 \quad v_1 \quad v_2$

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} w_1 \\ w_2 \\ v_1 \\ v_2 \end{array} \quad \begin{array}{l} w_1+w_2 \to w_2 \\ -2w_1+v_1 \to v_1 \\ -w_1+v_2 \to v_2 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \begin{array}{l} w_1 \\ w_1+w_2 \\ -2w_1+v_1 \\ -w_1+v_2 \end{array}$$

$2(w_1+w_2)-2w_1+v_1 \to -2w_1+v_1$

$w_1+w_2-w_1+v_2 \to -w_1+v_2$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{array}{l} w_1 \\ w_1+w_2 \\ 2w_2+v_1 \\ w_2+v_2 \end{array} \quad \begin{array}{l} -2w_2-v_1+w_2+v_2 \\ \to w_2+v_2 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} w_1 \\ w_1+w_2 \\ 2w_2+v_1 \\ -w_2-v_1+v_2 \end{array}$$

$\Downarrow$

$w_2 = -v_1+v_2$

$$\boxed{w_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}$$

$V+W = span\{\underbrace{(1,1,1,0)}_{w_1}, \underbrace{(-1,0,-1,0)}_{w_2}, \underbrace{(2,0,1,0)}_{v_1}\}$

$W \cap V = span\{\underbrace{(-1,0,-1,0)}_{w_2}\}$

4/4

QUESTION 6. Let $A = \begin{bmatrix} 2 & 4 & 2 \\ -2 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. Find the Smith-form of A over Z, i.e., Find invertible matrices R and C such that $RAC = D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where $|A| = \pm|D|$ and $a \mid b \mid c$.

$\to gcd = 2 \quad d_1 = \pm 2 \quad |A| = 4\begin{vmatrix} -2 & 2 \\ 0 & 2 \end{vmatrix} = 4(-4) = -16$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ -2 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1+R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -R_2+R_1 \to R_1$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C_1+C_3 \to C_3$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -C_2+C_3 \to C_3$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

4/4

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Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com